## PARAXIAL RAYTRACING

**Paraxial raytracing through multi–lens systems consisting of thin lenses**

The paraxial raytrace equations are an invaluable aid in determining many useful pieces of information about an optical system, including the following

**1)** Approximate ray heights at each lens. These are required to determine the clear diameters for each lens to avoid unacceptable light loss.

**2)** The focal lengths and positions of the principal planes of systems consisting of two or more elements.

**3)** The object and image positions, and the magnification of multi–element systems.

For further applications of paraxial raytracing, and for a more thorough introduction to the techniques, you may wish to consult one of the following books, which also discuss the tracing of rays outside the paraxial region

" Modern Optical Engineering" by W.J. Smith
" Elements of Optical System Design" by D.O'Shea

The parameters required are shown in the figure below. Here we use the following sign convention

Slope angles u are defined to be +ve if the ray slopes upwards to the right. Ray heights h are defined to be +ve if the ray is above the axis.

The parameters required are the separations $d_j$ between jth and (j+1)th component and the powers $K_j$ of the components.

[For thick lenses the values $d_j$ are separations between the second principal plane of the jth component and the first principal plane of the (j+1)th component.]

Successive application of the following equations, traces the path of the paraxial ray through the optical system.

$$u_{j+1} = u_j - h_j K_j$$

$$h_{j+1} = h_j + u_{j+1} d_{j+1}$$

for j=1...N where N is the number of lenses.

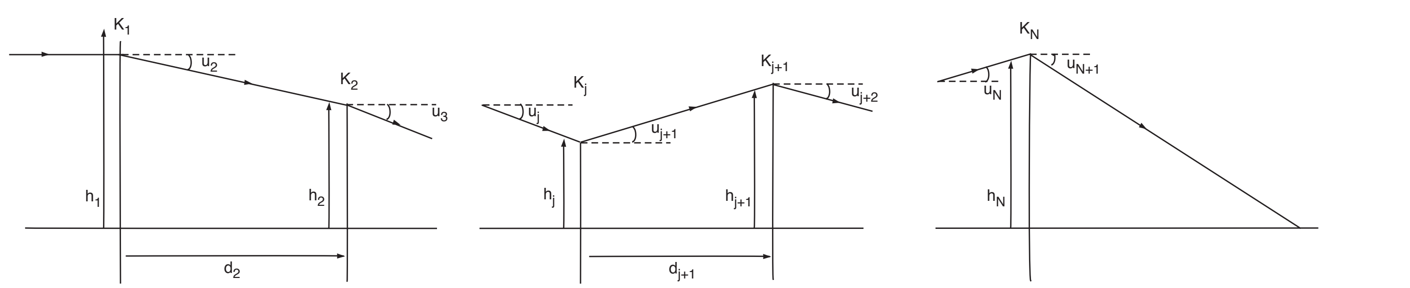
**Reverse Raytracing**

To trace rays in reverse, a simple manipulation of the last equations gives the procedure:

$$h_j = h_{j+1} - u_{j+1}\, d_{j+1}$$

$$u_j = u_{j+1} + h_j K_j$$

starting with $u_{N+1}$ and $h_N$ and working through until j=1




When using paraxial raytracing to examine an optical system, the 4 types of ray outlined here can often provide particularly useful information.

**Ray 1** A ray with $u_1 = 0$ and $h_1 = 1$. Tracing this ray allows the calculation of the focal length f and back focal distance $f_b$, from

$$f = -\frac{1}{u_{N+1}} \quad \text{and} \quad f_b = -\frac{h_N}{u_{N+1}}$$

**Ray 2** A ray with $u_{N+1}=0$ and $h_N=1$. Tracing this ray in reverse allows a check of the focal length f to be made and also gives the front focal distance $f_f$, from

$$f = \frac{1}{u_1} \quad \text{and} \quad f_f = \frac{h_1}{u_1}$$

**Ray 3** A ray traced for infinite object distance from the object position with $u_1$ set to some arbitrary value and $h_1 = -su_1$, where s is the object distance introduced earlier on Theory Pg 1. This ray allows the location of the image to be found, relative to the last lens, from

$$s' = -\frac{h_N}{u_{N+1}}$$

To obtain useful information about the ray heights and angles of the paraxial marginal ray, take either Ray 1 or Ray 3 for the case of an infinitely distant object or a finite object distance respectively. Then scale all $h_j$ and $u_j$ values so that the value of h at the stop is equal to the stop radius, or $|u_{N+1}|$ is equal to the output numerical aperture.
The magnification of the jth component is given by

$$m = \frac{u_j}{u_{j+1}}$$

**Ray 4** This ray should pass through the center of the aperture stop (the quantities have a 'bar' placed over them by convention) so that $\bar{h}=0$ at that surface. Either a forwards or backwards trace maybe carried out with an arbitrary value of $\bar{u}$ at the stop.

For an infinite object distance scale all the $\bar{u}$ and $\bar{h}$ values to make the initial field angle $\bar{u}$ correct.

For a finite object distance, determine the current object height of the arbitrary ray from $\bar{h}_1+s\bar{u}_1$ and then scale all h and u values until the object height is the desired value. Alternatively, one can work with the image height $\bar{h}_N+s'\bar{u}_{N+1}$, if that quantity has a target value.

A particularly useful feature of the linear nature of the paraxial equations is the ability to combine the paths of any two known paraxial rays to determine the path of another paraxial ray. Two uses are given here.

**Use of the paraxial raytrace to determine clear radii.**

To determine the required clear radii of components, the rays passing through the upper and lower edges of the aperture stop should be traced. In the absence of vignetting, the heights of these rays at surface j are given by $(\bar{h}_j+h_j)$ and $(\bar{h}_j-h_j)$ respectively. The required clear radius is the larger of these two in magnitude, that is $|\bar{h}_j|+|h_j|$.

**Use of the Paraxial raytrace to determine the range of angles present in radiation.**

The range of angles in any space can be found by forming the sum $|\bar{u}_j|+|\bar{u}_j|$. This can be useful in determining the optimum location to place components which have a strong angle sensitive behavior, such as Interference Filters and Polarizers.

**Lagrange Invariant H**

If we form the product

$$H = nu\eta$$

where n is the refractive index and u the paraxial marginal angle in object space and $\eta$ is the object height, then we find that this is equal to the corresponding product $n'u'\eta'$ formed from the parameters in image space.

The value H is known as the Lagrange invariant and has important consequences in many areas of optics.

When the object is at infinity, this form of the equation above becomes indeterminate in object space and is replaced by the alternative form

$$H = nh\theta$$

where h is the paraxial marginal ray height and $\theta$ is the paraxial chief ray angle in radians.

Additionally, H may be found in any space of an optical system given the refractive index n, and the paraxial marginal and chief ray parameters u, h, $\bar{u}$ and $\bar{h}$ from

$$H = n(u\bar{h}-\bar{u}h)$$

One implication of the Lagrange invariant is in illumination calculations especially the concentration of light onto fibers.

For example, given a collection cone angle and image patch size, the value of H is defined and no single channel optical system can improve collection efficiencies beyond a certain level. Increasing the solid angle of the radiation from the source automatically reduces the area of the source seen by the same amount. Of course, if the source is highly directional in output there may be changes in the total energy flux, but there will be a point of diminishing returns.

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**Example 5**

The system used in Example 3 (Theory Pg 4) can also be treated using the paraxial raytracing equations.

If we set an initial beam diameter of 15mm, this is a paraxial ray height $h_1$ of 7.5mm; also for an object at infinity $u_1=0$.

Following the procedure of the paraxial raytrace we obtain

$$
\begin{aligned}
u_2 &= u_1 - h_1K_1 \\
&= 0 - (7.5)(1/75) \\
&= -0.1 \\
h_2 &= h_1+d_2u_2 \\
&=7.5+37.5(-0.1) \\
&=3.75 \\
u_3 &=u_2-h_2K_2 \\
&=-0.1-3.75(1/75) \\
&=-0.15
\end{aligned}
$$

The focal length is given by

$$
\begin{aligned}
f &=-h_1/u_3 \\
&=-7.5/(-0.15)=50 \text{ mm}
\end{aligned}
$$

and the back focus by

$$
\begin{aligned}
f_b &=-h_2/u_2 \\
&=-3.75/(-0.15)=25 \text{ mm.}
\end{aligned}
$$

Once again the necessary adjustments must be made if the lenses are of finite thickness, by referring all distances to the appropriate principal points.

If we place the stop at the first lens with a field angle of 0.1 radians, then the values for the paraxial chief ray are as follows

$\bar{u}_1 = 0.1$
$\bar{h}_1 = 0$
$\bar{u}_2 = 0.1$
$\bar{h}_2 = 3.75$
$\bar{u}_3 = 0.05$

The image height $\eta$ is given by

$$\bar{h}_2+f_b\bar{u}_3 = 5\text{mm}$$

which is also equal to $f\theta$.

**Paraxial Raytracing and Gaussian Beams**

Another useful application is in determining the beam waist position and radius of a Gaussian beam. To do this, trace the following two rays from the input beam waist location, at distance $d_1$ in front of the input lens.

Ray 1 – $h_0=\omega_0$ (the $1/e^2$ radius of the beam at the input waist), $u_1=0$

Ray 2 – $\bar{h}_0=0$, $\bar{u}_1=\lambda/\pi\omega_0$ (where $\lambda$ is the wavelength). This is the far field semi–divergence angle.

At any position, the beam diameter is given by

$$(h^2+\bar{h}^2)^{fi}$$

The distance of output beam waist relative to the appropriate principal plane of lens j can be found from

$$\boxed{\frac{-(h_ju_{j+1}+\bar{h}_j\bar{u}_{j+1})}{u^2_{j+1} + \bar{u}^2_{j+1}}}$$

The radius of that waist is given by

$$\boxed{\omega_j= \frac{\lambda}{\pi(u^2_{j+1}+\bar{u}^2_{j+1})^{fi}}}$$

and the far field semi angle of divergence by

$$\boxed{\theta_j = (u^2_{j+1}+\bar{u}^2_{j+1})^{fi}}$$

**Example 6**

To compute the position and radius of the beam waist produced by a 50mm focal length lens, when used with a Helium–Neon laser with a beam waist diameter of 1.2mm located 25mm in front of the first principal plane of the lens.

In this case

Ray 1 – $h_0=0.6$mm, $u_1=0$

Ray 2 – $\bar{h}_0=0$mm

$$
\begin{aligned}
\bar{u}_1 &= \frac{0.6328\text{x}10^{-3}}{\pi\ 0.6} \\
&=3.357\text{x}10^{-4}\text{rads.}
\end{aligned}
$$

(the farfield semi–divergence angle)

At the lens, using $h_1 = h_0+d_1u_1$

$$
\begin{aligned}
h_1 &=0.6 \\
\bar{h}_1 &=(3.357\text{x}10^{-4})25 \\
&=8.393\text{x}10^{-3}\text{mm}
\end{aligned}
$$

and following refraction, using $u_2=u_1-h_1K_1$

$$
\begin{aligned}
u_2 &=-0.012\text{rads.} \\
\bar{u}_2 &=1.678\text{x}10^{-4}\text{rads.}
\end{aligned}
$$

Using the appropriate equations in the previous column, we find the output waist is located 49.98mm from the second principal plane of the lens with a beam waist radius of 16.78$\mu$m

For this particular case, beam waist location is not greatly shifted from the paraxial back focus. However this is not always the case with Gaussian beams, especially where the initial divergence is large.